On the Values for Factor Complexity

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Outline

Notation: Strings and languages

Finite Alphabet $X = \{0, \dots, r-1\}$, cardinality $|X| = r$

Finite strings (words) $w = x_1 \cdots x_n \in \{0,1\}^*$, $x_i \in \{0,1\}$

Length $|w| = n$

Languages $W \subseteq X^*, T \subseteq \{w : |w| = n\}$

Infinite strings (ω-words) $\xi = x_1 \cdots x_n \cdots \in X^\omega$

Prefixes of infinite strings $\xi[0..n] \in X^*$, $\big|\xi[0..n]\big| = n$

ω-Languages $F \subseteq X^\omega$

Subword (factor) complexity

Definition (Asymptotic subword complexity)

$$\tau(\xi) := \limsup_{n\to\infty} \frac{\log_r |\mathbf{infix}(\xi)\cap X^n|}{n}$$

$$\mathbf{infix}(\xi)\cap X^{n+m} \subseteq (\mathbf{infix}(\xi)\cap X^n)\cdot(\mathbf{infix}(\xi)\cap X^m)$$

Fact

The limit exists and equals $\tau(\xi) = \inf\left\{\frac{\log_r |\mathbf{infix}(\xi)\cap X^n|}{n} : n\in\mathbb{N}\right\}$.

Fact

$0 \le \tau(\xi) \le 1$ *and* $\mathbf{infix}(\xi) = X^*$ *if and only if* $\tau(\xi) = 1$.

Problem

Problem

For which $t, 0 \le t \le 1$, there is a $\xi \in X^\omega$ with $\tau(\xi) = t$?

Lemma (CAI/HARTMANIS '94)

For the (asymptotic) KOLMOGOROV *complexity* $\underline{\kappa} : X^\omega \to [0,1]$ *the following holds:*
For every $t, 0 \le t \le 1$, *there is a* $\xi \in X^\omega$ *with* $\underline{\kappa}(\xi) = t$.

Lemma (*Staiger '93*)

Let $W \subseteq X^*$ *be a regular language. Then there is a* $\xi \in X^\omega$ *with*

$$\tau(\xi) = \mathbf{H}_W := \limsup_{n\to\infty} \frac{\log_r(1 + |W \cap X^n|)}{n}.$$

Main theorem: One-dimensional case

Theorem

Let $\alpha \in [0,1]$. *Then there is an* ω*-language* $F \subseteq X^{\omega}$ *closed in the Cantor topology and computable in* α *such that*

1. $\dim F = \alpha$,
2. $\tau(\xi) = \alpha$ *for all* $\xi \in F$, *and*
3. *there is a* $\xi \in F$ *such that* $\underline{\kappa}(\xi) = \alpha$.

If, moreover, α *is a right-computable real number then* $\mathbf{pref}(F)$ *can be chosen to be computable.*

Main theorem: Multi-dimensional case

Definition (Multi-dimensional infixes)

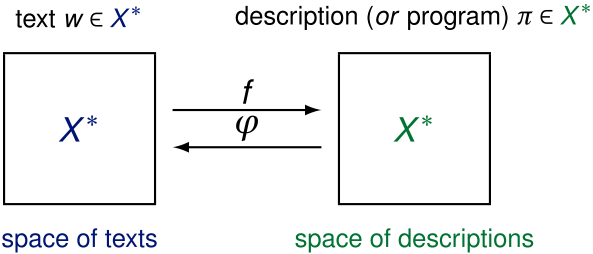
Let $\xi \in X^{\mathbb{N}^d}$. A *multi-dimensional subword* of ξ is a hypercube B of size n^d of ξ.

$$\tau_d(\xi) := \limsup_{n\to\infty} \frac{\log_r \left|\{B : B \in \textbf{infix}_d(\xi) \wedge \textbf{size}(B) = n^d\right|}{n}$$

Theorem

Let $\alpha \in [0,1]$ and $d \in \mathbb{N}, d \geq 2$. Then, given a decreasing sequence of rationals $(q_i)_{i\in\mathbb{N}}$ converging to α, there is an algorithm which constructs a nonempty set $M \subseteq X^{\mathbb{N}^d}$ such that $\tau_d(\xi)$ for all $\xi \in M$.

Compression: The principle of loss-less compression



text $w \in X^*$ description (*or* program) $\pi \in X^*$

$$\begin{tikzcd} X^* \arrow[r, bend left, "f"] & X^* \arrow[l, bend left, "\varphi"] \end{tikzcd}$$

space of texts space of descriptions

f is injective and $\varphi(f(w)) = w$ for all $w \in X^*$

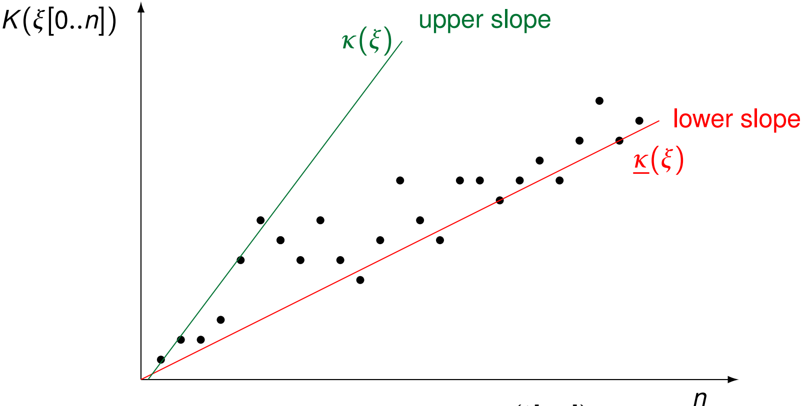
Complexity of w w.r.t. φ: $K_\varphi(w) := \inf\{|\pi| : \varphi(\pi) = w\}$

Fact (Combinatorial lower bound)

If $W \subseteq X^*$ has at least m elements then there is a $w \in W$ such that $K_\varphi(w) \geq \log_r m$.

Complexity of infinite words

Plot of the function $K(\xi[0..n])$



$$\underline{\kappa}(\xi) := \liminf_{n\to\infty} \frac{K(\xi[0..n])}{n}$$

Also known as *constructive dimension*.

KOLMOGOROV complexity and subword complexity

Theorem (SOLOMONOFF '64, KOLMOGOROV '65, CHAITIN '66)

There is an optimal partial-recursive function φ such that for all partial-recursive functions ψ there is a constant c_ψ such that

$$\forall w(w \in X^* \to K_\varphi(w) \leq K_\psi(w) + c_\psi\,.$$

Lemma (KOLMOGOROV '65)

If φ is an optimal partial-recursive function and $\underline{\kappa}(\xi) = \liminf_{n\to\infty} \frac{K_\varphi(\xi[0..n])}{n}$ *then* $\underline{\kappa}(\xi) \leq \tau(\xi)$.

X^ω as CANTOR space

Metric: $\rho(\eta,\xi) := \inf\{r^{-|w|} : w \in \mathbf{pref}(\eta) \cap \mathbf{pref}(\xi)\}$

Balls: $w \cdot X^\omega = \{\eta : w \in \mathbf{pref}(\eta)\} = \{\eta : w \sqsubset \eta\}$

Diameter: $\operatorname{diam} w \cdot X^\omega = r^{-|w|}$

$\operatorname{diam} F = \inf\{r^{-|w|} : F \subseteq w \cdot X^\omega\}$

Open sets: $W \cdot X^\omega = \bigcup_{w \in W} w \cdot X^\omega$

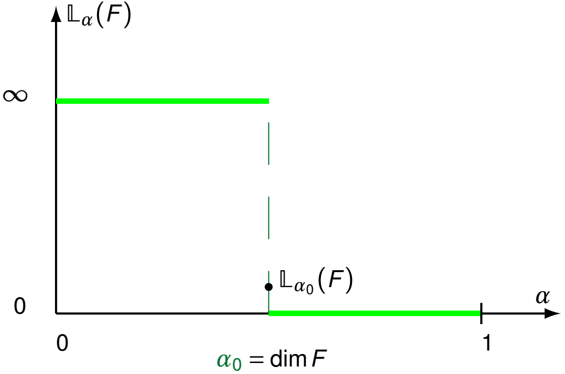
Closure: (Smallest closed set containing F)
$\mathscr{C}(F) = \{\xi : \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$

Fact

$F \subseteq X^\omega$ is closed if and only if $\mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)$ implies $\xi \in F$.

HAUSDORFF dimension: Definition

$$\mathbb{L}_\alpha(F) := \lim_{n\to\infty} \inf\left\{ \sum_{v\in V} r^{-\alpha\cdot|v|} : F \subseteq \bigcup_{v\in V} v\cdot X^\omega \wedge \min_{v\in V}|v| \geq n \right\}$$



$$\dim F := \inf\{\alpha : \mathbb{L}_\alpha(F) = 0\} = \sup\{\alpha : \mathbb{L}_\alpha(F) = \infty\}$$

HAUSDORFF dimension: Properties

Fact

1. dim *is monotone and countably stable:*
   $\dim \bigcup_{i \in \mathbb{N}} F_i = \sup\{\dim F_i : i \in \mathbb{N}\}$, *and* $\dim\{\xi\} = 0$
2. *If* $T \subseteq X^\ell$ *then* $\dim T^\omega = \dfrac{\log_{|X|} |T|}{\ell}$.

Theorem (Mass distribution principle)

Let μ *be a measure on* X^ω *such that* $\mu(F) > 0$ *and suppose that for some* α *there are numbers* $c_0 > 0$ *and* $n_0 \in \mathbb{N}$ *such that*

$$\forall w \big(w \in X^* \wedge n_0 \leq |w| \to \mu(w \cdot X^\omega) \leq c_0 \cdot (r^{-|w|})^\alpha\big).$$

Then $\mathbb{L}_\alpha(F) \geq \mu(F)/c_0$.

HAUSDORFF dimension: Relations to τ

Lemma (RYABKO '86,*Staiger '93*)

If $F \subseteq X^{\omega}$ *in not empty then* $\dim F \leq \sup\{\tau(\xi) : \xi \in F\}$.

Lemma (*Staiger '93*)

If $T \subseteq X^{*}$ *is finite then* $\dim T^{\omega} = \sup\{\tau(\xi) : \xi \in F\}$ *and* $\tau(\xi) = \dim T^{\omega}$ *for some* $\xi \in T^{\omega}$.

An auxiliary theorem

Theorem

Let $\alpha \in (0,1)$. Then there is an ω-language $F \subseteq X^{\omega}$ closed in the Cantor topology such that

1. *F has non-null α-dimensional measure $\mathbb{L}_{\alpha}(F)$.*
2. *$\tau(\xi) = \alpha$ for all $\xi \in F$.*
3. **pref**(F) *is computable in α.*

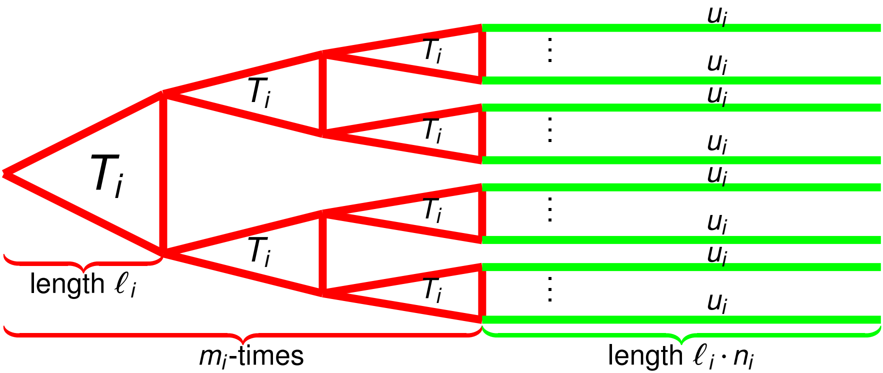
If, moreover, α is a right-computable real number then **pref**(F) *is computable.*

Proof.

Construct F as the limit of spherically symmetric trees $T_i \subseteq X^{\ell_i}$. □

The construction of T_{i+1}

$$T_{i+1} := T_i^{m_i} \cdot u_i \text{ where } u_i \in T_i^{n_i} \text{ and } \tau(u_i) \supseteq T_i$$



Properties of T_{i+1}

Start with $T_0 := X$, let $T_{i+1} := T_i^{m_i} \cdot u_i$, where $u_i \in T_i^*$ and

$$F := \{\xi : \xi \in X^\omega \wedge \mathbf{pref}(\xi) \subseteq \bigcup_{i \in \mathbb{N}} \mathbf{pref}(T_i)\}.$$

Then

1. $T_{i+1} \subseteq T_i^{m_i + n_i}$,
2. $T_i \subseteq X^{\ell_i}$, where $\ell_0 = 1$ and $\ell_{i+1} = m_i \cdot \ell_i + n_i \cdot \ell_i$, that is,
3. $\ell_i = \prod_{j=0}^{i-1} (m_j + n_j)$,
4. $|T_i| = \prod_{j=0}^{i-1} m_i$, and consequently,
5. $\dim T_i^\omega = \frac{\log_{|X|} |T_i|}{\ell_i} = \frac{m_0}{m_0 + n_0} \cdots \frac{m_{i-1}}{m_{i-1} + n_{i-1}}$.
6. $F \subseteq \bigcap_{i \in \mathbb{N}} T_i^\omega$, and consequently,
7. $\dim F \leq \inf\{\dim T_i^\omega : i \in \mathbb{N}\}$

Properties of F

$$F := \{\xi : \xi \in X^\omega \wedge \mathbf{pref}(\xi) \subseteq \bigcup_{i\in\mathbb{N}} \mathbf{pref}(T_i)\}$$

The following holds, for T_i and $\ell \leq \ell_i$:

Inclusion: $\mathbf{pref}(T_i) \subseteq \mathbf{pref}(F)$,

Extension: $\mathbf{pref}(T_i) \cap X^\ell = \mathbf{pref}(T_{i+1}) \cap X^\ell$, and

Spherical symmetry: $\mathbf{pref}(T_i) \cap X^\ell = (\mathbf{pref}(T_i) \cap X^{\ell-1}) \cdot X$, or

$$\left|\mathbf{pref}(T_i) \cap X^\ell\right| = \left|\mathbf{pref}(T_i) \cap X^{\ell-1}\right|$$

Conclusion

① $\mathbf{pref}(F) = \bigcup_{i\in\mathbb{N}} \mathbf{pref}(T_i)$

② $\mu(w \cdot X^\omega) := \begin{cases} 1/|\mathbf{pref}(F) \cap X^{|w|}|, & \text{if } w \in \mathbf{pref}(F)\text{, and} \\ 0, & \text{otherwise.} \end{cases}$

defines a measure on X^ω.

The rôle of α

Let $1 > q_0 > q_1 > \cdots > q_i > \cdots > \alpha = \lim_{i\to\infty} q_i$ where $q_i \in \mathbb{Q}$.
Then choose n_i and m_i in such a way that

$$n_i \geq |T_i| \text{ and } q_i = \frac{m_0}{m_0+n_0} \cdots \frac{m_i}{m_i+n_i},$$

that is,

$$\begin{aligned} q_0 &= \frac{m_0}{m_0+n_0} \quad \text{, and} \\ q_i/q_{i-1} &= \frac{m_i}{m_i+n_i}. \end{aligned}$$

Finally define u_i as a product of n_i words $w \in T_i$ (including all) in some (computable) order.

Conclusion

1. $\dim F \leq \alpha \leq q_i = \dim T_i^\omega$
2. since $\tau(u_i) \supseteq T_i$, we have $\tau(\xi) = \alpha$ for $\xi \in F$.

Dimension and KOLMOGOROV complexity

Lemma

The measure μ *satisfies* $\mu(w\cdot X^\omega)\le r^{-\alpha\cdot|w|}$ *for all* $w\in X^*$.

Application to F **via *Mass Distribution Principle***

F has HAUSDORFF dimension $\dim F=\alpha$ and $\mathbb{L}_\alpha(F)>0$

Theorem (*Staiger '93*)

Let $E\subseteq X^\omega$ *with* $\mathbb{L}_\alpha(E)>0$ *and let* $\varphi: X^*\to X^*$ *be a partial function and* $f:\mathbb{N}\to\mathbb{N}$ *such that* $\sum_{n\in\mathbb{N}} r^{-f(n)}<\infty$. *Then*

$$\exists\xi(\xi\in E\wedge\forall^\infty n(K_\varphi(\xi[0..n])\ge\alpha\cdot n-f(n)))\,.$$

Application to F

There is a $\xi\in F$ such that $\underline{\kappa}(\xi)=\alpha$.

Concluding remarks

On relative computability

For the proofs of lower bounds for the complexity K_φ we use only combinatorial bounds. Thus relativisation to computability in α is given.

To obtain upper bounds, observe that $K_\psi(w) \le K_\varphi(w) + c$ when φ is an optimal partial-recursive function and ψ is an optimal partial-recursive function relative to a certain oracle M.

The construction of F is constructive in any decreasing sequence of rationals $(q_i)_{i\in\mathbb{N}}$ converging to α.

On other complexities

In the theorem we can replace $\underline{\kappa}$ by other complexities (upper KOLMOGOROV complexity κ or finite-state dimensions).

Thank you