

There are Forty Nine Kuratowski Lattices

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Topological spaces

Definition (Topological spaces)

Let X be a non-empty set. The pair (X, \mathcal{O}) is called a **topological space** provided

- 1 $\mathcal{O} \subseteq 2^X, X \in \mathcal{O}$,
- 2 \mathcal{O} is closed under arbitrary union, and
- 3 \mathcal{O} is closed under finite intersection.

Remark

The family \mathcal{O} is usually called the family of *open* subsets of the space X .

The complements of open sets are referred to as *closed*.

Topological Closure and Interior [KURATOWSKI 1922]

Interior

- 1 $I\mathcal{X} = \mathcal{X}$
- 2 $IIM = IM \subseteq M$
- 3 $I(M_1 \cap M_2) = IM_1 \cap IM_2$

Closure

- 1 $\mathbf{C}\emptyset = \emptyset$
- 2 $\mathbf{C}CM = \mathbf{C}M \supseteq M$
- 3 $\mathbf{C}(M_1 \cup M_2) = \mathbf{C}M_1 \cup \mathbf{C}M_2$

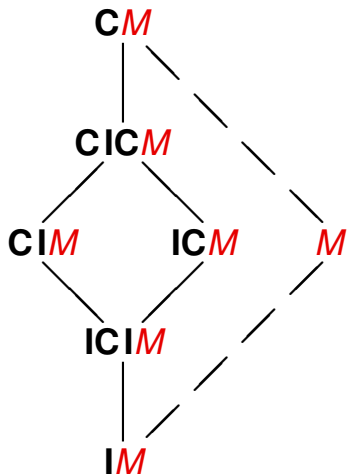
Fact (Duality)

$$\mathbf{C}M = \mathcal{X} \setminus I(\mathcal{X} \setminus M)$$

Theorem

If \mathbf{C} is a closure operator and I is the corresponding interior then $\mathbf{C}I\mathbf{C}IM = \mathbf{C}IM$ and $I\mathbf{C}I\mathbf{C}IM = I\mathbf{C}M$, for every $M \subseteq \mathcal{X}$.

Kuratowski lattice



No Kuratowski lattice



Hand - Verband, anti - isomorph zu einem
Teilverband eines Unterarm - Verbandes.

from W. HEISE and P. QUATTROCCHI
Informations- und Codierungstheorie, Springer, Berlin 1983

Kuratowski lattices: Examples

Co-dense dense sets

Definition

A set M is *dense (co-dense)*
 $\iff \mathbf{C}M = X \ (\mathbf{C}(X \setminus M) = X)$

$$\mathbf{C}M = \mathbf{C}\mathbf{C}M = \mathbf{I}M (= X)$$

|
 M
|

$$\mathbf{C}I M = \mathbf{I}C I M = I M (= \emptyset)$$

Non-closed nowhere dense sets

Definition

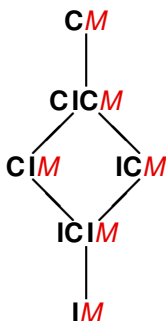
A set M is *nowhere dense*
 $\iff \mathbf{I}C M = \emptyset \iff \mathbf{C}I C M = \emptyset$

$\mathbf{C}M$

|
 M
|

$$\mathbf{C}I C M = \dots = I M (= \emptyset)$$

Possible inclusions between M and its derived sets



lower bounds		upper bounds	
(A_0)	$\mathbf{CM} = M$	(A_1)	$\mathbf{IM} = M$
(B_0)	$\mathbf{CICM} \subseteq M$	(B_1)	$\mathbf{ICIM} \supseteq M$
(C_0)	$\mathbf{CIM} \subseteq M$	(C_1)	$\mathbf{ICM} \supseteq M$
(D_0)	$\mathbf{ICM} \subseteq M$ $\mathbf{ICM} = \mathbf{IM}$	(D_1)	$\mathbf{CIM} \supseteq M$ $\mathbf{CIM} = \mathbf{CM}$
(E_0)	$\mathbf{ICIM} \subseteq M$ $\mathbf{ICIM} = \mathbf{IM}$	(E_1)	$\mathbf{CICM} \supseteq M$ $\mathbf{CICM} = \mathbf{CM}$
relations between derived sets			
(F_0)	$\mathbf{CICM} = \mathbf{ICM}$	(F_1)	$\mathbf{ICIM} = \mathbf{CIM}$
(G)	$\mathbf{ICM} \subseteq \mathbf{CIM}$ $\mathbf{CICM} = \mathbf{CIM}$	$\mathbf{ICIM} = \mathbf{ICM}$	
(H)	$\mathbf{CIM} \subseteq \mathbf{ICM}$		

How many Kuratowski lattices exist

2^{14} combinations of $(A_0), \neg(A_0), \dots, (H), \neg(H)$



some relations between $(A_0), \dots, (H)$

+

computer search



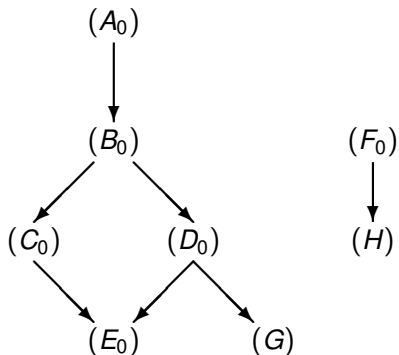
49 combinations (types)



Verification of their existence in CANTOR space

General Implications (Relations)

(A_0)	$\mathbf{CM} = M$
(B_0)	$\mathbf{CICM} \subseteq M$
(C_0)	$\mathbf{CIM} \subseteq M$
(D_0)	$\mathbf{ICM} \subseteq M$
(E_0)	$\mathbf{ICIM} \subseteq M$
(F_0)	$\mathbf{CICM} = \mathbf{ICM}$
(G)	$\mathbf{ICM} \subseteq \mathbf{CIM}$
(H)	$\mathbf{CIM} \subseteq \mathbf{ICM}$



Further Implications

Proposition

Let $\alpha \in \{0, 1\}$. Then the following implications hold true.

$$C_0 \wedge C_1 \Rightarrow H$$

$$F_{1-\alpha} \Rightarrow (E_\alpha \leftrightarrow C_\alpha)$$

$$G \Rightarrow (E_\alpha \leftrightarrow D_\alpha) \wedge (B_\alpha \leftrightarrow C_\alpha)$$

$$G \wedge H \Rightarrow (E_\alpha \leftrightarrow B_\alpha \wedge C_\alpha \wedge D_\alpha) \wedge F_0 \wedge F_1 \wedge (E_0 \wedge E_1 \leftrightarrow A_0 \wedge A_1)$$

The case $\neg G \wedge \neg H$

type	A_0	A_1	B_0	B_1	C_0	C_1	D_0	D_1	E_0	E_1	F_0	F_1	G	H
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2a	0	0	0	0	0	0	0	0	1	0	0	0	0	0
2b	0	0	0	0	0	0	0	0	0	1	0	0	0	0
3a	0	0	0	0	1	0	0	0	1	0	0	0	0	0
3b	0	0	0	0	0	1	0	0	0	1	0	0	0	0
4	0	0	0	0	0	0	0	0	1	1	0	0	0	0
5a	0	0	0	0	1	0	0	0	1	1	0	0	0	0
5b	0	0	0	0	0	1	0	0	1	1	0	0	0	0

$$\neg G \Rightarrow \bigwedge_{\alpha=0}^1 \neg A_\alpha \wedge \neg B_\alpha \wedge \neg D_\alpha$$

$$\neg H \Rightarrow \neg F_0 \wedge \neg F_1$$

The case $\neg G \wedge H$

type	A_0	A_1	B_0	B_1	C_0	C_1	D_0	D_1	E_0	E_1	F_0	F_1	G	H
6	0	0	0	0	0	0	0	0	0	0	0	0	0	1
7a	0	0	0	0	0	0	0	0	1	0	0	0	0	1
7b	0	0	0	0	0	0	0	0	0	1	0	0	0	1
8a	0	0	0	0	1	0	0	0	1	0	0	0	0	1
8b	0	0	0	0	0	1	0	0	0	1	0	0	0	1
9	0	0	0	0	0	0	0	0	1	1	0	0	0	1
10a	0	0	0	0	1	0	0	0	1	1	0	0	0	1
10b	0	0	0	0	0	1	0	0	1	1	0	0	0	1
11	0	0	0	0	1	1	0	0	1	1	0	0	0	1
12a	0	0	0	0	0	0	0	0	0	0	1	0	0	1
12b	0	0	0	0	0	0	0	0	0	0	0	1	0	1
13a	0	0	0	0	0	0	0	0	1	0	1	0	0	1
13b	0	0	0	0	0	0	0	0	0	1	0	1	0	1
14a	0	0	0	0	1	0	0	0	1	0	1	0	0	1
14b	0	0	0	0	0	1	0	0	0	1	0	1	0	1
15a	0	0	0	0	0	1	0	0	0	1	1	0	0	1
15b	0	0	0	0	1	0	0	0	1	0	0	1	0	1
16a	0	0	0	0	0	1	0	0	1	1	1	0	0	1
16b	0	0	0	0	1	0	0	0	1	1	0	1	0	1
17a	0	0	0	0	1	1	0	0	1	1	1	0	0	1
17b	0	0	0	0	1	1	0	0	1	1	0	1	0	1
18	0	0	0	0	0	0	0	0	0	0	1	1	0	1
19a	0	0	0	0	1	0	0	0	1	0	1	1	0	1
19b	0	0	0	0	0	1	0	0	0	1	1	1	0	1
20	0	0	0	0	1	1	0	0	1	1	1	1	0	1

The case $G \wedge \neg H$

type	A_0	A_1	B_0	B_1	C_0	C_1	D_0	D_1	E_0	E_1	F_0	F_1	G	H
21	0	0	0	0	0	0	0	0	0	0	0	0	1	0
22a	0	0	0	0	0	0	1	0	1	0	0	0	1	0
22b	0	0	0	0	0	0	0	1	0	1	0	0	1	0
23a	0	0	1	0	1	0	1	0	1	0	0	0	1	0
23b	0	0	0	1	0	1	0	1	0	1	0	0	1	0
24a	1	0	1	0	1	0	1	0	1	0	0	0	1	0
24b	0	1	0	1	0	1	0	1	0	1	0	0	1	0
25	0	0	0	0	0	0	1	1	1	1	0	0	1	0
26a	1	0	1	0	1	0	1	1	1	1	0	0	1	0
26b	0	1	0	1	0	1	1	1	1	1	0	0	1	0

$$G \Rightarrow (E_\alpha \leftrightarrow D_\alpha) \wedge (B_\alpha \leftrightarrow C_\alpha)$$

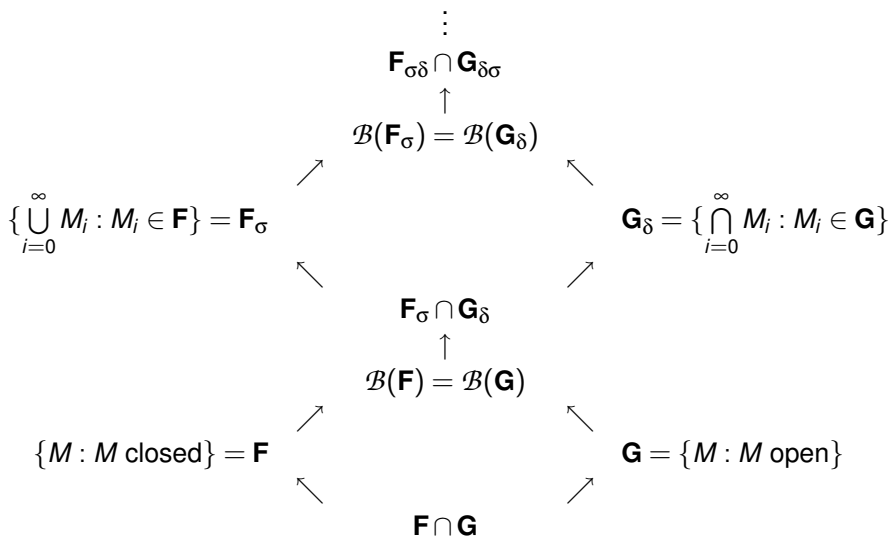
$$\neg H \Rightarrow \neg F_0 \wedge \neg F_1$$

The case $G \wedge H$

type	A_0	A_1	B_0	B_1	C_0	C_1	D_0	D_1	E_0	E_1	F_0	F_1	G	H
27	0	0	0	0	0	0	0	0	0	0	1	1	1	1
28a	0	0	1	0	1	0	1	0	1	0	1	1	1	1
28b	0	0	0	1	0	1	0	1	0	1	1	1	1	1
29a	1	0	1	0	1	0	1	0	1	0	1	1	1	1
29b	0	1	0	1	0	1	0	1	0	1	1	1	1	1
30	1	1	1	1	1	1	1	1	1	1	1	1	1	1

$$G \wedge H \Rightarrow (E_\alpha \leftrightarrow B_\alpha \wedge C_\alpha \wedge D_\alpha) \wedge F_0 \wedge F_1 \wedge (E_0 \wedge E_1 \leftrightarrow A_0 \wedge A_1)$$

Topological complexity: BOREL hierarchy



Topological complexity: Theorems on lower bounds

Theorem (On condition (G))

Let X be a topological space. The family $\mathcal{G} := \{M : M \subseteq X \wedge \mathbf{ICM} \subseteq \mathbf{CIM}\}$ is a Boolean algebra which contains all open (and closed) and nowhere dense subsets of X .
If, moreover, X is a metric space then \mathcal{G} contains all sets in $\mathbf{F}_\sigma \cap \mathbf{G}_\delta$.

Theorem (On conditions (C_α) and (A_α))

Let X be a topological space, $E \subseteq X$ open and $F \subseteq X$ closed.

- 1 If $M = E \cup F$ and $\mathbf{CIM} \subseteq M$ then M is closed.
- 2 If $M = E \cap F$ and $\mathbf{ICM} \supseteq M$ then M is open.

Topological complexity: lower bounds

Lemma

- 1 M is of type 30 if and only if $M \in \mathbf{G} \cap \mathbf{F}$.
- 2 M is of type 24a, 26a or 29a if and only if $M \in \mathbf{F} \setminus \mathbf{G}$.
- 3 M is of type 24b, 26b or 29b if and only if $M \in \mathbf{G} \setminus \mathbf{F}$.

Corollary

- 1 A set of type 1, ..., 20 cannot be in $\mathcal{G} \supseteq \mathcal{B}(\mathbf{G})$.
- 2 A set of type 21, 22a, 22b, 25 or 27 cannot be in $\mathbf{G} \cup \mathbf{F}$.
- 3 A set of type 23a or 28a cannot be a union of an open and a closed set.
- 4 A set of type 23b or 28b cannot be an intersection of an open and a closed set.

Verification: Examples in CANTOR space

Definition (CANTOR space)

The **CANTOR space** is the metric space $(2^{\mathbb{N}}, \rho)$ where for $\xi, \eta \subseteq \mathbb{N}$

$$\rho(\xi, \eta) := \sup\{2^{-n} : n \in \mathbb{N} \wedge n \in \xi \Delta \eta\}.$$

Basis

- 1 Let $\Gamma \subseteq \{0, \dots, n\}$ and define $B_{n, \Gamma} := \{\xi : \xi \cap \{0, \dots, n\} = \Gamma\}$.
- 2 The family $\{B_{n, \Gamma} : \Gamma \subseteq \{0, \dots, n\} \wedge n \in \mathbb{N}\}$ is a basis of the CANTOR space.
- 3 Moreover, all sets $B_{n, \Gamma}$ are simultaneously open and closed, and $\mathbf{G} \cap \mathbf{F}$ consists of all finite unions of sets of the form $B_{n, \Gamma}$.

Verification: Duality

Proposition (Self-Duality and Duality)

- 1 *Condition (G) or (H) hold for M if and only if it holds for its complement $X \setminus M$.*
- 2 *Let $Z \in \{A, B, C, D, E, F\}$. Condition (Z_0) holds for M if and only if condition (Z_1) holds for $X \setminus M$.*

Proposition

- 1 *Let $\tau \in \{1, 4, 6, 9, 11, 18, 20, 21, 25, 27, 30\}$. The set M is of type τ if and only if $X \setminus M$ is of type τ .*
- 2 *Let $\tau \in \{1, 2a, 2b, \dots, 30\} \setminus \{1, 4, 6, 9, 11, 18, 20, 21, 25, 27, 30\}$. The set M is of type τa if and only if $X \setminus M$ is of type τb .*

Verification: Composition

Definition (Shift and mark)

Let $\xi \in 2^{\mathbb{N}}$.

$$\text{sm}_0(\xi) := \{0\} \cup \{n+1 : n \in \xi\}$$

$$\text{sm}_1(\xi) := \{n+1 : n \in \xi\}$$

$$\text{sm}_0(M_0) \cup \text{sm}_1(M_1) = \{\text{sm}_0(\xi) : \xi \in M_0\} \cup \{\text{sm}_1(\xi) : \xi \in M_1\}$$

Proposition

Let M_0, M_1 be subsets of the CANTOR SPACE $(2^{\mathbb{N}}, \rho)$.

If M_0 is of type $(\alpha_1, \alpha_2, \dots, \alpha_{14})$ and M_1 is of type $(\beta_1, \beta_2, \dots, \beta_{14})$ then $\text{sm}_0(M_0) \cup \text{sm}_1(M_1)$ is of type $(\alpha_1 \wedge \beta_1, \alpha_2 \wedge \beta_2, \dots, \alpha_{14} \wedge \beta_{14})$.

Conclusion

It is sufficient to prove that the types 16a, 17a, 20, 26a, 28a, 29a, and 30 do exist in CANTOR space.

Verification: Examples in $\mathbf{FO}[\mathbb{N}, <]$

$$16a \quad M_{16} := \{ \xi : \xi \neq \emptyset \wedge (\min \xi + 1 \in \xi \vee (\min \xi + 1 \notin \xi \wedge \forall t \exists t' (t < t' \wedge t' \in \xi))) \}$$

$$17a \quad M_{17} := M_{16} \cup \{ \emptyset \}$$

$$20 \quad M_{20} := \{ \xi : \forall t \exists t' (t < t' \wedge t' \in \xi) \}$$

and all co-dense dense sets

$$28a \quad M_{28} := \{ \xi : \exists t (t \in \xi \wedge \forall t' (t' \in \xi \rightarrow t = t')) \}$$

and all non-closed nowhere dense sets

$$29a \quad M_{29} := M_{28} \cup \{ \emptyset \}$$

and all non-empty closed nowhere dense sets

$$30 \quad M_{30} := \emptyset$$

and all clopen sets

Remark: Upper bounds

It is possible to construct examples in $\mathbf{FO}[\mathbb{N}, <]$ of lowest topological complexity.

Connected spaces

A topological space X is connected provided \emptyset and X are the only clopen subsets, e.g. $X = \mathbb{R}$.

Theorem

In a connected space there are no sets satisfying

$$\neg(G) \wedge (F_0) \wedge \neg(E_1): \quad \mathbf{ICIM} \subset \mathbf{ICM} = \mathbf{CICM} \subset \mathbf{CM},$$

$$\neg(E_0) \wedge (F_1) \wedge \neg(G): \quad \mathbf{IM} \subset \mathbf{ICIM} = \mathbf{CIM} \subset \mathbf{CICM}, \text{ or}$$

$$\neg(E_0) \wedge (G) \wedge (H) \wedge \neg(E_1): \mathbf{IM} \subset \mathbf{ICIM} \subseteq \mathbf{ICM} = \mathbf{CIM} \subseteq \mathbf{CICM} \subset \mathbf{CM}$$

Conclusion

In a connected space there are no sets of type 12a, 12b, 13a, 13b, 14a, 14b, 18, 19a, 19b, or 27.

Theorem

The remaining 39 types exist in the real line \mathbb{R} .