A Polynomial-Time Algorithm for the Automatic Baire Property

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Computability in Europe
Trier, July 31, 2026

Notation: Strings and Languages

Finite Alphabet $X = \{0, \ldots, r-1\}$, cardinality $|X| = r$

Finite strings (words) $w = x_1 \cdots x_n \in X^*$, $x_i \in X$

Length $|w| = n$

Languages $W \subseteq X^*$

Infinite strings (ω-words) $\xi = x_1 \cdots x_n \cdots \in X^\omega$

Prefixes of infinite strings $\xi[0..n] \in X^*$, $|\xi[0..n]| = n$

ω-Languages $F \subseteq X^\omega$

X^ω as CANTOR space

Balls: $w \cdot X^\omega = \{\eta : w \in \mathbf{pref}(\eta)\} = \{\eta : w \sqsubset \eta\}$

Open sets: $W \cdot X^\omega = \bigcup_{w \in W} w \cdot X^\omega$

Closure (Smallest closed set containing F):

$$\mathscr{C}(F) = \{\xi : \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$$

Fact

$F \subseteq X^\omega$ is closed if and only if $\mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)$ implies $\xi \in F$.

Cantor Space: Density

Definition (Nowhere dense)

$F \subseteq X^\omega$ is nowhere dense in X^ω if and only if $\mathscr{C}(F)$ does not contain a ball $w \cdot X^\omega$.

Fact (Exit property)

$F \subseteq X^\omega$ is nowhere dense in X^ω if and only if for all $w \in \mathbf{pref}(F)$ there is a $v_w \in X^*$ such that $w \cdot v_w \notin \mathbf{pref}(F)$.

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$F \subseteq X^\omega$ is of first Baire category (or meagre) in X^ω if and only if F is a countable union of nowhere dense sets.

Definition (Baire property)

$F \subseteq X^\omega$ has the Baire property if and only if there are an open set $W \cdot X^\omega$ and a meagre set F' such that $F \,\Delta\, W \cdot X^\omega = F'$.

Behaviour of ω-automata

- Consider a deterministic automaton $\mathscr{A} = (X, S, \delta, s_0)$.
- When an infinite word $\xi = x_1 x_2 \cdots$ is read by the automaton it induces an infinite run, an infinite sequence of states: $s_{i+1} = \delta(s_i, x_{i+1})$

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- $$\begin{array}{ccccccccccccc} s_0 & & s_1 & & s_2 & & s_{i-1} & & s_i & & \\ & \searrow & \uparrow & \searrow & \uparrow & \cdots & \uparrow & \searrow & \uparrow & \searrow & \cdots \\ & & x_1 & & x_2 & & x_{i-1} & & x_i & & \end{array}$$

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- This run is summarised by
$$\mathrm{Inf}_{\mathscr{A}}(\xi) = \{s : \exists^\infty i (s = s_i)\},$$
the set of states visited infinitely often by the run.

Acceptance of ω-languages by ω-automata

- Muller acceptance condition: a set of subsets of states (the table) $\mathscr{T} \subseteq 2^S$
- An infinite word ξ is (Muller-)accepted if $\mathrm{Inf}_{\mathscr{A}}(\xi) \in \mathscr{T}$.

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Definition (Regular ω-language)

An ω-language $F \subseteq X^\omega$ is called *regular* if and only if F is accepted by a finite automaton, that is, $F = \{\xi : \mathrm{Inf}_{\mathscr{A}}(\xi) \in \mathscr{T}\}$ for some automaton $\mathscr{A}$ and table $\mathscr{T}$.

Deterministic Büchi automata

Theorem (Büchi '62)

The family of regular ω-languages over X is closed under Boolean operations.

- Complements of ω-languages accepted by deterministic Büchi automata are not necessarily accepted by deterministic Büchi automata [Landweber '69].

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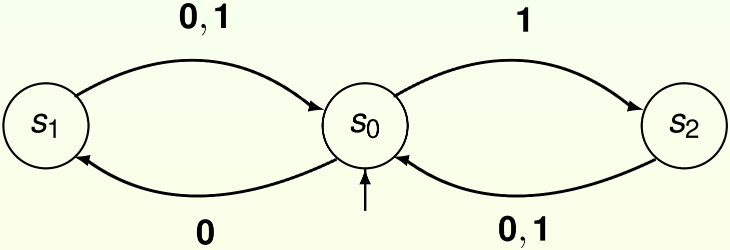
- Not all regular ω-languages are accepted by deterministic Büchi automata [Landweber '69].
- An ω-language $F \subseteq X^{\omega}$ is accepted by deterministic Büchi automata if and only for all Muller automata $\mathscr{A}$ and tables $\mathscr{T}$ with $F = L(\mathscr{A}, \mathscr{T})$ the set $\mathscr{T} \cap \mathrm{LOOP}_{\mathscr{A}}$ is upwards closed w.r.t. $\subseteq$. [*St.* and Wagner '74, Wagner '79]

An automaton for $F_1 = ((\{0,1\}^2)^* \cdot 1\{0,1\}0\{0,1\})^\omega$

$\xi \in F_1 \iff \xi$ contains infinitely many **0**s and **1**s at odd places



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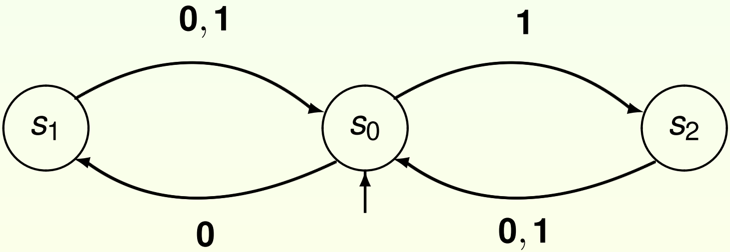

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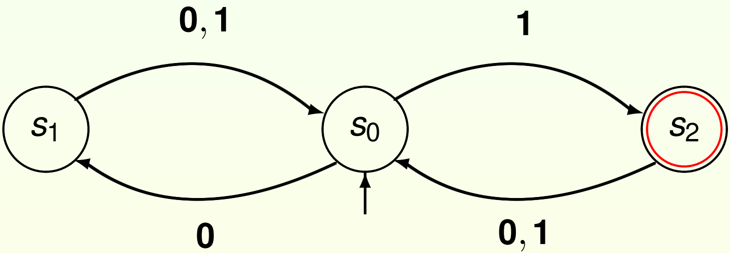

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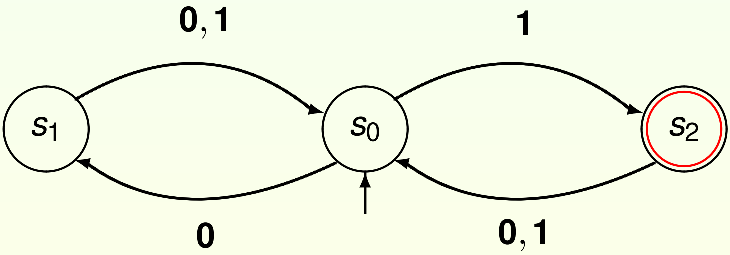

With Muller condition: Use the table $\mathscr{T} = \{\{s_0, s_2\}, \{s_0, s_1, s_2\}\}$

Loops and strongly connected components

Consider the automaton $\mathscr{A} := (X, S, s_0, \delta)$ as a (multi-)graph.

Loop (Strongly connected subset) $Z \in \mathrm{LOOP}_{\mathscr{A}}$:

All $s, s' \in Z$ are likewise reachable.

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Terminal strongly connected component $Z \in \mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}}$:

A maximal loop in $\mathscr{A}$ ***without*** exit edge.

Loops and density

Property

Let $L_n(\mathscr{A}, Z) := \{\xi : \forall w (w \sqsubset \xi \wedge |w| \geq n \to \delta(s_0, w) \in Z)\}$. *Then*

1. *If* $Z \in \mathrm{LOOP}_{\mathscr{A}} \setminus \mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}}$ *then* $L_n(\mathscr{A}, Z)$ *is nowhere dense.*
2. *If* $Z \in \mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}}$ *then* $L_n(\mathscr{A}, Z) = \{w : |w| = n \wedge \delta(s_0, w) \in Z\} \cdot X^{\omega}$ *is open.*

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   Moreover $\bigcup_{n \in \mathbb{N}} L_n(\mathscr{A}, Z) = \{\xi : \mathrm{Inf}_{\mathscr{A}}(\xi) \subseteq Z\}$, *and, consequently,*
3. *if* $Z \in \mathrm{LOOP}_{\mathscr{A}} \setminus \mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}}$ *then* $\{\xi : \mathrm{Inf}_{\mathscr{A}}(\xi) \subseteq Z\}$ *is of first Baire category, and*
4. *if* $Z \in \mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}}$ *then* $\{\xi : \mathrm{Inf}_{\mathscr{A}}(\xi) \subseteq Z\}$ *is open.*

Automatic Baire property [Finkel'20]

Definition (Finkel'20)

$F \subseteq X^\omega$ has the Automatic Baire property if and only if there are an open regular ω-language $W \cdot X^\omega$ and a meagre regular ω-language F' such that F' is a countable union of closed sets (a Σ_2-set) and

$$F \,\Delta\, W \cdot X^\omega \subseteq F' \,.$$

Theorem (Finkel'20,'21)

Every regular ω-language fulfils the Automatic Baire property.

Automatic Baire property [Proof]

For an automaton $\mathscr{A} := (X, S, s_0, \delta)$ and a table $\mathscr{T} \subseteq \mathrm{LOOP}_{\mathscr{A}}$ we have

$$\mathscr{T} = (\mathscr{T} \cap \mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}}) \cup (\mathscr{T} \setminus \mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}})$$

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$$L(\mathscr{A}, \mathscr{T}) \,\Delta\, L\overbrace{\left(\mathscr{A}, \left\{Z' : \exists Z(Z \in \mathscr{T} \cap \mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}} \wedge Z' \subseteq Z)\right\}\right)}^{\text{open}}$$

$$\subseteq L\underbrace{(\mathscr{A}, 2^S \setminus \mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}})}_{\text{meagre}}$$

Construction of Muller Automata

- As complementation of Muller automata might lead to an exponential blow-up in the size of the table $\mathscr{T}$, we use

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- Given an automaton $\mathscr{A} := (X, S, s_0, \delta)$ and a table $\mathscr{T} \subseteq 2^S$ construct Muller automata $\mathscr{A}_{\mathrm{open}}$ and $\mathscr{A}_{\mathrm{meagre}}$ and tables $\mathscr{T}_{\mathrm{open}}$ and $\mathscr{T}_{\mathrm{meagre}}$ such that

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- construct Büchi automata $\mathscr{B}_{\mathrm{open}}$ and $\mathscr{B}_{\mathrm{meagre}}$ and sets of final states T_{open} and T_{meagre} such that

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Automata for $\mathscr{U} = \{Z' : \exists Z(Z \in \mathscr{T} \cap \mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}} \wedge Z' \subseteq Z)\}$

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If *all* loops of of a *terminal* SCC Z are accepting (rejecting) this SCC Z can be merged into *one* accepting (rejecting) state s_Z:

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If *all* loops of of a *terminal* SCC Z are accepting (rejecting) this SCC Z can be merged into *one* accepting (rejecting) state s_Z:

$$Z \in \mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}} \to s_Z$$

Transform $(\mathscr{A}, \mathscr{T})$ into $(\mathscr{A}_{\mathrm{open}}, \mathscr{T}_{\mathrm{open}})$, or $(\mathscr{B}_{\mathrm{open}}, T_{\mathrm{open}})$, respectively:

- Merge $Z \in \mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}}$ into s_Z with $\delta_{\mathrm{open}}(s_Z, x) := s_Z$.
- Change all transitions $\delta(s, x) = z \in Z \in \mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}}$ to $\delta_{\mathrm{open}}(s, x) := s_Z$.
- The other parts of $\mathscr{A}$ remain unchanged: $\mathscr{A}_{\mathrm{open}} = \mathscr{B}_{\mathrm{open}}$.
- For Muller automata $\mathscr{T}_{\mathrm{open}} := \{\{s_z\} : Z \in \mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}}\}$
- For Büchi automata $T_{\mathrm{open}} := \{s_z : Z \in \mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}}\}$

Muller automaton for $\mathscr{V} = 2^S \setminus \mathrm{SCC}^{\mathsf{t}}_{\mathscr{A}}$

$$L(\mathscr{A}, 2^S \setminus \mathrm{SCC}^{\mathsf{t}}_{\mathscr{A}}) = X^\omega \setminus L(\mathscr{A}, \mathrm{SCC}^{\mathsf{t}}_{\mathscr{A}})$$

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- The size of the table $\mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}}$ is polynomial in the size of $\mathscr{A}$.
- $(\mathscr{A}, \mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}})$ has size polynomial in the size of $\mathscr{A}$.

Büchi automaton for $\mathscr{V} = 2^S \setminus \mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}}$, Theorem

Theorem

Let $\mathscr{A} = (X; S; s_0; \delta)$ be an automaton and $\mathscr{T} \subseteq 2^S$ be a table such that $\mathscr{T} \cap \mathrm{LOOP}_{\mathscr{A}} \subseteq \mathrm{SCC}_{\mathscr{A}}$. Then there are a deterministic Büchi automaton $\mathscr{A}' = (X, S', \delta', s_0')$ with $O(|S|^2)$ states and a $T' \subseteq S'$ such that $L(\mathscr{A}, \mathscr{T}) = L(\mathscr{A}'; T')$.

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Idea of proof

Simulate $\mathscr{A}$ in the first component, and in the second check in a *previously given order* whether **all** states of a $Z \in \mathscr{T} \cap \mathrm{SCC}_{\mathscr{A}}$ appear.

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Idea of proof

Simulate $\mathscr{A}$ in the first component, and in the second check in a *previously given order* whether **all** states of a $Z \in \mathscr{T} \cap \mathrm{SCC}_{\mathscr{A}}$ appear. If one $Z \in \mathrm{SCC}_{\mathscr{A}}$ is left, it can never be entered again.

Büchi automaton for $\mathscr{V} = 2^S \setminus \mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}}$

$$L(\mathscr{A}, 2^S \setminus \mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}}) = X^{\omega} \setminus L(\mathscr{A}, \mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}})$$

- The table $\mathrm{SCC}^{\mathbf{t}}_{\mathscr{A}}$ satisfies the conditions of the theorem.
- Construct $\mathscr{B}_{\mathrm{meagre}}$ and T_{meagre} according to the theorem.

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4. Compare $\mathrm{SCC}^t_{\mathscr{A}}$ with $\mathscr{T}$ to obtain $\mathrm{SCC}^t_{\mathscr{A}} \cap \mathscr{T}$.
   $$\mathrm{time}_4 = \mathrm{poly}(\mathrm{size}(\mathscr{A}) + \mathrm{size}(\mathscr{T}))$$

Vielen Dank für Ihre Aufmerksamkeit

Thank you for your attention